

Random fields and probability distributions with given marginals on randomly correlated systems: a general method and a problem from theoretical neuroscience

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2000 J. Phys. A: Math. Gen. 33 23

(<http://iopscience.iop.org/0305-4470/33/1/302>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.118

The article was downloaded on 02/06/2010 at 08:01

Please note that [terms and conditions apply](#).

Random fields and probability distributions with given marginals on randomly correlated systems: a general method and a problem from theoretical neuroscience

Carlo Fulvi Mari[†]

SISSA Programme in Neuroscience, International School for Advanced Studies, Trieste, Italy

E-mail: C.Fulvi-Mari@lboro.ac.uk

Received 29 March 1999

Abstract. A class of families of marginal probabilities on sets of discrete random variables is studied and a necessary and sufficient condition for the consistency of the given marginals is provided. This result allows one to verify the consistency of the marginals through a Boltzmann statistical analysis.

The procedure is then applied in order to verify the hypotheses assumed in a recent model of neocortical associative areas, according to which connected modules of neurons are simultaneously active with probability higher than chance, and inter-modular connections are very diluted. The verification becomes a typical problem of extremely diluted spin systems in Boltzmann–Gibbs ensemble. The results presented here justify the assumptions made in the neuroscientific theory, and an upper bound to the inter-modular activity correlation is found.

1. Introduction and summary

The probability distribution of a proper subset from the set of random variables (RVs) of a stochastic system is usually called *marginal probability* or simply *marginal* and from elementary probability it can be obtained by integration of the joint probability distribution function (PDF) over the values of the RVs that are not involved in the marginal to be calculated.

When the complete PDF is not known and use is made of an Ansatz for its marginals, the problem of verifying the consistency of such marginals arises. In other words, it is necessary to demonstrate the existence of at least one joint PDF whose marginals coincide with the conjectured ones. Otherwise, the probabilistic model would be unrealizable: no system of RVs that follow the required statistics could be defined. In the case of a physical theory, such inconsistency would mean that no real system could be described by that theory (as long as probability theory is able to model physical systems).

The problem of the consistency of a set of marginals belongs to the mathematical theory of probability distributions with given marginals. Some works about this subject can be found in the mathematics literature, the most remarkable ones being, perhaps, the papers by Vorobev [1] in a combinatorial context, and by Kellerer [2] in a more functional measure-theoretic fashion.

In this paper I deal with the problem of marginal consistency over sets of discrete RVs whose pair correlations are randomly assigned, so that the system may be represented as a random graph whose nodes can be each in one of several possible states.

[†] Present address: Nonlinear and Complex Systems Group, Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, UK.

In section 2, I propose a general systematic method for verifying the consistency of a family of marginals of a certain class, defined over stochastic systems of discrete RVs, making use of results from the theory of random fields (RFs) and of Vorobev's theorem. In sections 3–5, I deal with a specific marginal problem to which I partially apply the method. In section 3, I describe the neuroscientific origins of the specific problem mainly concerning the activity distribution in mammalian neocortex. In section 4, I define this problem in terms of a probabilistic model. I put in evidence the underlying question of probability distribution with given marginals, and show the inapplicability of the already existing theorems on consistency by Vorobev and Kellerer. I also derive some conditions on the parameters of the model from an assumption of self-averaging. In section 5, approximating the method introduced in section 2 through an Ansatz, I create a fictitious stochastic dynamics on the configuration space of the system, with the aim of obtaining an asymptotic distribution of the activities that satisfies the marginals and which, at the same time, can be studied through numerical simulations. Indeed, as I show, the analytical approach to the study of the asymptotic distribution allows the verification of the consistency of the given marginals only in a particular limit, due to difficulties in calculations. Then, I describe and discuss the results of the numerical simulations, that favour the hypothesis of consistency of the given family of marginals, thus supporting the underlying neuroscientific theory.

In section 6, I summarize the results.

2. Mathematical basis of the method

Let S be a finite set of N discrete random variables, and \mathcal{M} a set of marginals such that any marginal is defined either over a pair of RVs or on a single RV from S . For any marginal $\mathcal{P}_A \in \mathcal{M}$ over a RV or a pair of RVs, called A , in S , I call A the *support* of \mathcal{P}_A . The marginals of all the possible pairs in S are assumed to be present in \mathcal{M} .

The following procedure allows one to transform the complicated verification of the consistency of \mathcal{M} into a statistical Boltzmann problem. This mapping is mainly based on two steps: (1) using Vorobev's theorem, the set of marginals is transformed; (2) it is observed that this new set of PDFs is consistent if and only if they constitute a random field. At that point, the representation theorem of RFs and Gibbs states can be applied.

Compatibility. The marginals in \mathcal{M} have to be compatible, that is the marginalization of every pair of them on the intersection of their supports must coincide. Otherwise, \mathcal{M} is a set of inconsistent marginals. For example, let $F(x_1, x_2)$ and $G(x_2, x_3)$ be two of the marginals in \mathcal{M} ; then it is necessary that

$$\sum_{x_1} F(x_1, x_2) = \sum_{x_3} G(x_2, x_3)$$

for any value of x_2 . In the rest I assume the compatibility of the marginals in \mathcal{M} .

Construction of the RF. Given a variable $x_i \in S$, one may consider all the pair marginals in \mathcal{M} whose supports contain x_i . This subfamily is denoted \mathcal{N}_i , and Y_i is the set of RVs that belong to the supports of \mathcal{N}_i . The RVs in $Y_i - \{x_i\}$ are the *neighbours* of x_i from the point of view of the Markov RF theory. The marginals in \mathcal{N}_i are compatible and their supports constitute a regular complex [3]. Thus Vorobev's theorem [1] can be applied: *At least one joint probability distribution of all the variables in Y_i exists such that its marginals over the supports of \mathcal{N}_i coincide with the given marginals in \mathcal{N}_i .* Any distribution with this property is called an *extension* of \mathcal{N}_i .

Usually there are several extensions. Suppose one has two extensions $\mathcal{P}_{\mathcal{N}_i}^{(1)}$ and $\mathcal{P}_{\mathcal{N}_i}^{(2)}$ of \mathcal{N}_i . Then for any $x_m \in Y_i - \{x_i\}$:

$$\sum_{x'} \mathcal{P}_{\mathcal{N}_i}^{(1)}(x_i, x_m, x') = \sum_{x'} \mathcal{P}_{\mathcal{N}_i}^{(2)}(x_i, x_m, x') \quad (1)$$

where x' is a collective symbol for $\{x_j \in Y_i - \{x_i, x_m\}\}$ and the sums are over the values they can assume. It follows that the two extensions can differ only by a function $F_i = \mathcal{P}_{\mathcal{N}_i}^{(1)} - \mathcal{P}_{\mathcal{N}_i}^{(2)}$ such that

$$\sum_{x'} F_i(x_i, x_m, x') \equiv 0 \quad (2)$$

for every RV $x_m \in Y_i - \{x_i\}$. Thus, knowing one particular extension \mathcal{P}_i of \mathcal{N}_i , all the other extensions of \mathcal{N}_i can be obtained from \mathcal{P}_i by adding functions F_i defined on Y_i satisfying the property of equation (2).

Considering only binary RVs ($x_k = 0, 1$), the function F_i can always be written as

$$F_i(x) = \sum_{\{m,n \neq i\}} (1 - 2x_m)(1 - 2x_n) \psi_{mn}(x_{Y_i - \{x_m, x_n\}}) \quad (3)$$

where x represents all the variables in Y_i , the sum is over the indices of the RVs in $Y_i - \{x_i\}$, and ψ_{mn} may depend on all the RVs in $Y_i - \{x_m, x_n\}$. Thus one can choose a convenient extension of \mathcal{N}_i and then add to it the generic function in equation (3). As a convenient extension I take the conditional independence distribution

$$\mathcal{P}_i^{(0)}(x) = \mathcal{P}(x_i) \prod_A \mathcal{P}_A(x_A | x_i) \quad (4)$$

where the product is over all the supports of \mathcal{N}_i , \mathcal{P}_A is derived from the marginal in \mathcal{N}_i with support A , and x_A is the other RV that belongs to the support A together with x_i . The distribution in equation (4) has the property that its marginalization over any variable in $Y_i - \{x_i\}$ is still a conditional (with respect to x_i) independence distribution.

Thus, the most general joint PDF on binary RVs in Y_i that satisfies the marginals in \mathcal{N}_i can be written as

$$\mathcal{P}_i(x_{Y_i}) = \mathcal{P}_i^{(0)}(x_{Y_i}) + \sum_{\{m,n \neq i\}} (1 - 2x_m)(1 - 2x_n) \psi_{mn}(x_{Y_i - \{x_m, x_n\}}) \quad (5)$$

where x_R is the collective symbol for all the RVs in the set $R \subseteq \mathcal{S}$. Since ψ_{mn} additively modifies the probabilities of the events related to Y_i , it cannot be completely arbitrary, so that its values can move only in bounded intervals.

For simplicity I indicate the local solutions by Ψ_i only, implicitly keeping the arbitrariness, provided by equation (5), that represents degrees of freedom possibly useful to the solution of the marginal problem.

From the parametrized joint distribution Ψ_i , it is easy to extract the conditional probability $\Phi_i(x_i | \{x_k \in Y_i, k \neq i\})$. (For simplicity, I assume that no event has null probability.)

Performing this construction for every variable x_m in \mathcal{S} , one obtains the field $\{\Phi_i, i = 1, \dots, N\}$ that is a RF if and only if the marginals are consistent.

Derivation of the Gibbs field. From the theorem of representation of RFs and Gibbs fields [4] it follows that, given a RF, a Gibbs field that generates the RF exists with the canonical potential

$$\begin{aligned} V_X(\eta) &= \sum_{U \subseteq X} (-1)^{|X-U|} \ln \Phi_i(\eta^U) \quad \forall i \in X \in \mathcal{C} \\ &= 0 \quad \forall X \notin \mathcal{C} \end{aligned} \quad (6)$$

where η is an event (that is, a realization of the RVs in X), \mathcal{C} is the set of all the cliques of the RF, U is a subset of the clique X , η^U is the vector such that $\eta_k^U = \eta_k$ when $x_k \in U$ and $\eta_k^U = 0$ otherwise, and i indicates any variable (= node) in X .

Thus the Gibbsian distribution \mathcal{G}_S can be constructed[†], whose marginals possibly generate the RF.

Verification. At this point, possibly making use of the above-mentioned degrees of freedom, one has to verify whether all the marginals in \mathcal{M} can be obtained marginalizing \mathcal{G}_S . If they can, then the family \mathcal{M} is consistent and \mathcal{G}_S gives the solution (or the set of solutions, since there could remain some free parameters). If they cannot, then the given family \mathcal{M} of marginals is inconsistent. In a single proposition:

The family \mathcal{M} of compatible marginals is a consistent set of marginals if and only if the constructed Gibbs distribution \mathcal{G}_S is compatible with all those marginals.

This verification rule may be formalized and shown in a more rigorous treatment, and its hypotheses a little relaxed, but it would not be very useful in the present context.

The only problem now is provided by the practical calculations, where, in the case of large systems, statistical mechanics of Gibbs ensembles may be of some help. In fact, sometimes calculations cannot be performed easily, as in the case analysed in the rest of this paper, where the difficulty in managing with a large set of randomly correlated random variables does not allow one to apply the complete procedure presented in this section.

3. Neuroscientific origin of the problem

The primate neocortex is a tissue containing a very large number of interacting neurons that receive inputs from, and send outputs to, subcortical systems. One of the functions performed by the neocortex, especially in its associative areas (frontal, parietal and inferotemporal), could be the cued retrieval of information stored during past experiences, that is the recovering of certain meaningful neuronal activity distributions (*patterns*). This hypothesis is supported by anatomical findings according to which recurrent connectivity is common in the neocortex [5], and it is theoretically well known that recurrent networks are able to retrieve stored patterns (see, e.g., Hopfield [6], one of the precursors in the field). In particular, neurons seem to be grouped into *modules* [7] in which neurons are densely and recurrently interconnected. The connectivity among neurons belonging to different modules is also recurrent but very sparse. Other evidence from electrophysiological experiments [8–10] have revealed locally persistent activity immediately following memory retrieval, a phenomenon that can be explained with models of recurrent neuronal networks [11]. The local activity seems to be correlated to the modular structure of the connectivity.

The whole complex of modules could therefore be a large autoassociative memory, as proposed by Braitenberg [5]. The performance of such a network has been studied in [12] and [13], where the authors analysed a mathematical model of a recurrent network with modular structure and calculated its *capacity* (that is, the number of patterns that the network is able to store in its synapses) as a function of some constructive parameters. They consider each module as a fully connected network, and distribute the *long-range* connections of its neurons

[†] The Hamiltonian of the Gibbs distribution is

$$H(\omega) = \frac{1}{\beta} \sum_Q V_Q(\omega)$$

where Q is any subset of \mathcal{S} , ω is any configuration of the system, and $\frac{1}{\beta}$ is the ‘temperature’.

randomly to all the other modules of the large autoassociator. They suggested that such a network is a poor, implausible model of biological cortices since it is affected by the presence of undesired stable activity states in which each module retrieves a fraction of a pattern, while different modules do not co-operate to retrieve the global pattern that the whole modular network has stored. Such a network would often produce a ‘puzzle’ of several original activity patterns.

In [14] we found a way to avoid these problems and thus to restore the possibility of a modular autoassociative network to plausibly model neocortical areas.

First, we introduced sparseness in the modular activity: Each module (m) is active ($\tau_m = 1$) in any pattern with probability $\tau \ll 1$. Then, we supposed that the long-range projections of a neuron are concentrated onto the neurons of a small number of other modules (about s'). This is in agreement with the experiments discussed in [5], where in the rat they found s' in the order of a few units. We abstracted these findings assuming that the long-range projections pass through what we call *channels* that connect any module to other modules randomly sorted out from the whole set during the ‘construction’ of the network, and then quenched. The randomness in the quenched channels is a first approximation of the biological structures that maybe derive from a combination of phylogenetic instructions and chance.

We then introduced statistical dependence of connected modules: if a module receives projections from another module that is active, its probability to be active is larger than chance ($t_1 > \tau$); unconnected modules are nearly independent. The set of natural patterns is such that many features are not independent. Our hypothesis is that the correlation among the features composing the natural patterns are reflected in the structure of the cortico–cortical connections. In fact, it would seem useless or even ruinous to build up connections between modules that analyse independent features in natural patterns, so that these modules cannot exchange useful information, for instance, for a retrieval task with each other. Moreover, we think that the structure and function of the channels could be related to semantic representation in the cortex.

The modifications we introduced suppress the noxious states while leaving the correct retrieval states almost unaltered. We found that the larger the ratio $\frac{t_1}{\tau}$, the better the suppression of the undesired states. Thus, the question arises as to whether the correlated statistics is consistent, and, if so, how large $\frac{t_1}{\tau}$ can be.

In the rest of this paper I define this mathematical problem of marginals in more general terms and find some results addressing its solution. In order to imagine the following in a neuronal context, the reader should identify the nodes of the random graph and their states as the modules and the modular activities, respectively. The presence of a channel between modules m and n will be represented by the binary connection variable s_{mn} being equal to 1, while it is equal to 0 when the two modules are not connected to each other. The channels are assumed to be bi-directional, that is $s_{mn} = s_{nm}$.

4. The randomly correlated system

Consider a set of binary RVs $\{\tau_m, m = 1, \dots, M\}$, with large M ; the configurations assumed by this set trial by trial are points in $\{0, 1\}^M$. Let the average ‘activity’ $\langle \frac{1}{M} \sum_m \tau_m \rangle$ be equal to τ . (Angle brackets indicate the ensemble average.)

The RVs $\{\tau_m\}$ are not independent. I consider the case in which the joint PDF $\mathcal{P}(\{\tau_m\})$ is unknown and an Ansatz is proposed on the pair marginals (as in the problem of section 3). To simplify the formulation, it is useful to give an intuitive representation of the system from the beginning.

Consider a random graph $G(M, s)$, where s is the probability for any edge to be present; the adjacency matrix (s_{mn}) is symmetric, that is the graph is undirected, and there is no simple

loop ($s_{mm} = 0$). Each node represents one of the RVs; the presence of the edge indicates a marked dependence in the activities of the two nodes, while two non-adjacent variables are nearly independent (the dependence in a pair is defined as the non-factorability of their marginal PDF[†]). To be more precise, the marginal distribution of the activities in a randomly chosen pair of nodes is given by the following probability table:

$$\begin{aligned}\mathcal{P}(\tau_m = 1, \tau_n = 1|s_{mn}) &= \tau t_1 s_{mn} + \tau^2 (1 - s_{mn}) \\ \mathcal{P}(\tau_m = 1, \tau_n = 0|s_{mn}) &= (1 - \tau) t_0 s_{mn} + (1 - \tau) \tau (1 - s_{mn})\end{aligned}\quad (7)$$

where t_1 is a probability larger than τ and t_0 is smaller than τ . According to equations (7), the probability that two connected nodes are in the same state of activity (0 or 1), averaged over the connected pairs, is greater than chance (positive dependence). Actually, if one looks at the same pair (m, n) during trials, the probabilities of the activities of the two nodes are

$$\begin{aligned}\mathcal{P}(\tau_m = 1, \tau_n = 1|S) &= f_{mn}^{11} \tau t_1 s_{mn} + h_{mn}^{11} \tau^2 (1 - s_{mn}) \\ \mathcal{P}(\tau_m = 1, \tau_n = 0|S) &= f_{mn}^{01} (1 - \tau) t_0 s_{mn} + h_{mn}^{01} (1 - \tau) \tau (1 - s_{mn})\end{aligned}\quad (8)$$

where S represents the knowledge of the adjacency matrix, and f_{mn}^{11} , h_{mn}^{11} , f_{mn}^{01} , h_{mn}^{01} are positive *structure factors* that take into account the fluctuations of the marginals across the node pairs and depend only on the quenched structure of the graph. The structure factors are normalized as follows:

$$\begin{aligned}\frac{2}{sM(M-1)} \sum_{(m,n)} f_{mn}^{11} s_{mn} &= 1 \\ \frac{2}{sM(M-1)} \sum_{(m,n)} f_{mn}^{01} s_{mn} &= 1 \\ \frac{2}{(1-s)M(M-1)} \sum_{(m,n)} h_{mn}^{11} (1 - s_{mn}) &= 1 \\ \frac{2}{(1-s)M(M-1)} \sum_{(m,n)} h_{mn}^{01} (1 - s_{mn}) &= 1\end{aligned}\quad (9)$$

where

$$s = \frac{2}{M(M-1)} \sum_{(m,n)} s_{mn}.\quad (10)$$

Thus, the average of equations (8) over the connected and unconnected pairs returns us to equations (7) for example,

$$\frac{2}{sM(M-1)} \sum_{(m,n)} s_{mn} \mathcal{P}(\tau_m = 1, \tau_n = 1|S) = \mathcal{P}(\tau_m = 1, \tau_n = 1|s_{mn} = 1) = \tau t_1.\quad (11)$$

I also assume that the marginal distribution of a single node does not depend on the structure of connections, that is

$$\mathcal{P}(\tau_m = 1|S) = \tau.\quad (12)$$

Indeed, I want to study the existence of distributions with marginals deviating from independence without introducing an *a priori* position-dependent distribution of single-node activity. From equations (8) and (12)

$$\mathcal{P}(\tau_m = 1|S) = [f_{mn}^{11} \tau t_1 + f_{mn}^{01} (1 - \tau) t_0] s_{mn} + [h_{mn}^{11} \tau^2 + h_{mn}^{01} (1 - \tau) \tau] (1 - s_{mn}) = \tau.\quad (13)$$

[†] This definition is convenient also from the neurophysiological point of view since in experiments usually only a small number of modules can be simultaneously monitored.

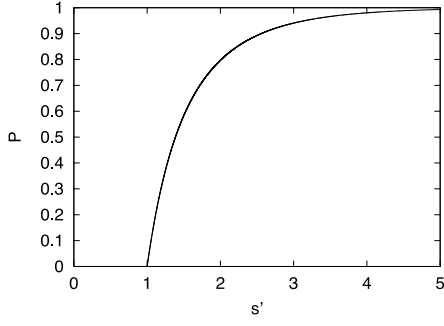


Figure 1. Probability for two nodes to be reachable one from the other through a path of the graph versus the average number of connections per node.

Then it must be that

$$\begin{aligned} f_{mn}^{11} \tau t_1 + f_{mn}^{01} (1 - \tau) t_0 &= \tau \\ h_{mn}^{11} \tau + h_{mn}^{01} (1 - \tau) &= 1. \end{aligned} \quad (14)$$

Averaging the first one over the connected pairs, I obtain the necessary condition

$$(1 - \tau) t_0 = \tau (1 - t_1). \quad (15)$$

This may be seen as an ‘equilibrium’ condition; in fact, if one averages equations (8) over the connected pairs and over the unconnected ones, respectively, one obtains

$$\begin{aligned} \mathcal{P}(\tau_m = 1 | s_{mn} = 1) &= \tau t_1 + (1 - \tau) t_0 \\ \mathcal{P}(\tau_m = 1 | s_{mn} = 0) &= \tau. \end{aligned} \quad (16)$$

Thus, the relation in equation (15) implies that the presence, or absence, of a connection between node m and any other node does not condition, by itself, the activity of node m (on average). This is consistent with equation (12).

Relation (15) may also be seen as a ‘detailed balance’: if each node represents the state of a dynamical system and this can move only from one state to an adjacent one at each time step, then the probability of observing the evolution of the system from state m of ‘class 1’ ($\tau_m = 1$) to state n of ‘class 0’ ($\tau_n = 0$) is equal to the probability of observing the reverse kind of transition.

One of the aims of this paper is to show that the marginals in equations (8) over the random graph can be realized if the parameters of this stochastic system satisfy appropriate conditions. In particular, I show that, for fixed s and τ , there exists an upper bound to t_1 above which the network can no longer support the desired statistics.

Trivially, the problem is solvable if $t_1 = \tau$, that is with independent RVs. I concentrate on the case with $t_1 > \tau$; the case with $t_1 < \tau$ is similarly treatable.

The probability for two nodes to be reachable, one from the other, through a path in $G(M, s)$ is easily found to be as in figure 1 versus the average number of connections per node ($= s' = s \cdot (M - 1)$, the analytical relation being $s' = -\frac{\ln(1-p)}{p}$; cf appendix A). In the range interesting for neuroscience ($2 < s' < 10$) such probability is very near to 1. If t_1 were equal to 1, the large majority of the nodes would be in the same state (0 or 1), which is not acceptable. Thus, t_1 must be smaller than 1.

Is it sufficient to keep t_1 smaller than 1? One is brought to conjecture that the higher the correlation inside the set of nodes (that is the higher s'), the lower the upper bound to t_1 . This idea seems to be confirmed by the results obtained in this paper.

4.1. Existing theorems

The theorem of Kellerer [2, 15] gives a necessary and sufficient condition for the consistency of a finite set of marginals, but in the present context it is not useful. In fact, since I am dealing with a large number of randomly correlated RVs, the hypotheses of Kellerer's theorem cannot be verified for each specific realization of the random graph representing the correlations. On the other hand, I could verify the hypotheses *in probability* when considering the thermodynamic limit ($M \rightarrow \infty$), but then I would have an infinite set of marginals and the cited theorem could not be applied. Besides, the theorem requires the verification of a condition over a large set of test functions, and this would also be a difficult task.

The crucial hypothesis in Vorobev's theorem [1] is the regularity of the complex of the subsets of RVs that are the arguments of the given marginals. Roughly speaking, the set of marginals is consistent (or 'extendable' in Vorobev's terminology) if it is not possible to describe a cycle in the cited complex passing from one subset to another one only if these have not empty intersection. Obviously, this is not the case in the present model, since every pair of nodes is argument of a marginal. So, Vorobev's theorem cannot be applied in the present context. However, it should be noted that most of the pair marginals are almost factorable, and this could be a hint toward an extension of Vorobev's theorem to the present model.

4.2. Conditions from self-averaging

A first condition on the parameters is obtained from the reasonable assumption that in the thermodynamic limit the average activity on the graph is not affected by the condition that one particular node is active (actually, this is equivalent to requesting self-averaging of the average activity $\frac{1}{M} \sum_m \tau_m$; cf appendix B):

$$\left\langle \frac{1}{M-1} \sum_{m \neq n} \tau_m \middle| \tau_n = 1 \right\rangle = \frac{1}{M-1} \sum_{m \neq n} \mathcal{P}(\tau_m = 1 | \tau_n = 1, S) \longrightarrow \tau \quad (17)$$

for any n . Then the average over the nodes must be

$$\frac{1}{M} \sum_n \frac{1}{M-1} \sum_{m \neq n} \mathcal{P}(\tau_m = 1 | \tau_n = 1, S) \longrightarrow \tau. \quad (18)$$

Using equations (8), it follows that

$$s \cdot (t_1 - \tau) = 0. \quad (19)$$

If s is finite when $M \rightarrow \infty$, t_1 cannot be greater than τ . Thus, the stochastic system I consider in this paper makes sense only if $\lim_{M \rightarrow \infty} s = 0$. Indeed, it is interesting to study the statistical properties of the model without conditioning them through adequate tuning of the input patterns statistics: The main question is how the structure of connections can reflect the correlations among the parallel contributes of the input signals, keeping the average activity constant.

If I allowed for anticorrelation between unconnected nodes, that is

$$\begin{aligned} \mathcal{P}(\tau_m = 1, \tau_n = 1 | s_{mn}) &= \tau t_1 s_{mn} + \tau x (1 - s_{mn}) \\ \mathcal{P}(\tau_m = 1, \tau_n = 0 | s_{mn}) &= (1 - \tau) t_0 s_{mn} + (1 - \tau) x' (1 - s_{mn}) \end{aligned} \quad (20)$$

with $x < \tau$ and $x' = \tau \frac{1-x}{1-\tau} > \tau$, the previous argument of self-averaging would give the bound (see appendix B)

$$x = \frac{\tau - s t_1}{1 - s}. \quad (21)$$

This implies that t_1 must never be larger than $\frac{\tau}{s}$. In particular, t_1 must be equal to τ if $s = 1$ (while x loses physical meaning).

Thus, marginals in (20) can satisfy the request of self-averaging also with finite s . In this paper I discuss only the case with $s = O(\frac{1}{M})$, and hence with $x = \tau$, which seems to be the most interesting case, at least in relation to the neuroscientific problem. Perhaps useful results for the finite s case could be obtained by also introducing correlations in the set of connections $\{s_{mn}\}$, such as that the probability for the connection (i, j) to exist is larger than chance if the connections (i, k) and (k, j) exist, for any k . This would also be interesting from the point of view of neuroscience since there is appreciable evidence that any two small neocortical areas are more likely connected with each other if another area is connected to both [16].

5. A fictitious dynamics from a Markov RF

I introduce a stochastic dynamics of the activity of interacting nodes to construct a probability distribution whose marginals are those in equations (7), when s' is finite. I want to underscore that this dynamics is purely fictitious: no particular real physical process is assumed to underlie the stochastic system. The desired PDF is given by the asymptotic distribution of the dynamical process, so demonstrating that at least one solution to the marginal problem exists.

First, guessing an Ansatz, I consider the marginals in equation (8) with $f_{mn}^{11} = f_{mn}^{01} = 1$ for any connected pair (m, n) , and insert them into equation (4) in order to obtain the conditional probability distribution of the activities of all the nodes directly connected to another one, the condition being the activity of the latter:

$$\begin{aligned} \mathcal{P}(\{\tau_n, \forall n | s_{mn} = 1\} | \tau_m = 1) &= t_1^{n_+} (1 - t_1)^{n_-} \\ \mathcal{P}(\{\tau_n, \forall n | s_{mn} = 1\} | \tau_m = 0) &= t_0^{n_+} (1 - t_0)^{n_-} \end{aligned} \quad (22)$$

where n_+ and n_- are the numbers of adjacent nodes in activity level 1 and, respectively, 0. The assumption of conditional independence in equations (22) as a first approximation is partially justified by the absence, with probability equal to 1, of direct connections among the $n_+ + n_-$ nodes connected with module m , and can be considered as the zero-th-order approximation of equation (5) for small ψ_{mn} . This approximation is adopted due to the difficulties in the analytical treatment (see the following), and in fact is a limit to the application of the method that instead requires the most general local probability distribution.

For a pair of unconnected nodes (m, n) that share a common neighbour, equations (22) give

$$h_{mn}^{11} = \tau \left(\frac{t_1}{\tau} \right)^2 + (1 - \tau) \left(\frac{t_0}{\tau} \right)^2. \quad (23)$$

At this point, the PDF for a node given the activities of all its neighbours can be found through Bayesian inversion. After some algebra, this can be written as

$$\mathcal{P}(\tau_m = 1 | \{\tau_n, \forall n | s_{mn} = 1\}) = \frac{1}{1 + \exp\{-\beta(n_+ - \lambda n_- + \mu)\}} \quad (24)$$

being

$$\beta = \ln \frac{t_1}{t_0} \quad \beta\lambda = \ln \frac{1 - t_0}{1 - t_1} \quad \beta\mu = \ln \frac{\tau}{1 - \tau}. \quad (25)$$

Equation (24) defines a Markov RF over the graph if the probabilities defined in equations (22) are consistent.

Then I use equation (24) to generate the discrete-time stochastic asynchronous dynamics, that can be recognized as a standard discrete-time heat-bath dynamics [17].

At each time step:

- choose a node randomly

- update its activity with probability given by equation (24).

Actually, numerical simulations have shown that the system is more stable (in simulations M must be finite) and reaches equilibrium more easily if the term μ in equation (24), which is mainly responsible for keeping the average activity in the net at the desired level, is modified as the instantaneous average activity deviates from τ . Thus, in place of equation (24) I consider the following expression:

$$\mathcal{P}(\tau_m = 1 | \{\tau_n, \forall n \neq m\}) = \frac{1}{1 + \exp\{-\beta(n_+ - \lambda n_- + \mu \frac{1}{\tau M} \sum_m \tau_m)\}}. \quad (26)$$

So the attractor with average activity τ is strengthened. When, at equilibrium, the average activity is equal to τ , expression (26) coincides with expression (24).

5.1. Analytical approach

The single-spin Markov dynamics generated by equation (26) satisfies detailed balance in the thermodynamic limit. Hence, during time evolution, the system tends to an equilibrium state. It can be shown, with standard methods of the heat-bath literature, that the corresponding asymptotic PDF over the configurations is Boltzmann-like with Hamiltonian

$$H = -\frac{1-\lambda}{2} \sum_{i,j} s_{ij} \tau_i \tau_j + \lambda \sum_{i,j} s_{ij} \tau_i (1 - \tau_j) - \frac{\mu}{2\tau M} \left(\sum_i \tau_i \right)^2. \quad (27)$$

Applied to the Markov RF of equation (24), the representation theorem (section 2; [4, 18]) confirms, through equation (6), the form of the Hamiltonian in equation (27) (except for the strengthening factor of μ) in the thermodynamic limit, since the contributions to H by the cliques with more than two nodes are negligible. However, the theorem cannot substitute the construction of a dynamics since the latter is needed for the simulations until an analytical solution of the statistical canonical model with H is found (from which to extract the relation between t_1^{\max} and s'). It seems interesting that the use of the characteristics[†] of a Markov RF as updating laws for an asynchronous dynamics drives the system toward an equilibrium distribution that coincides with the global distribution provided by the equivalence theorem.

The Boltzmann distribution with the Hamiltonian of equation (27) is the candidate for the distribution with the desired marginals I am looking for. Mathematical tools of the statistical mechanics of the Boltzmann–Gibbs ensemble may be used, in principle, to study the equilibrium properties of the system. Unfortunately, the application of the mean-field techniques to the calculation of the partition function soon finds a serious mathematical obstacle that would require the introduction of a large number of order parameters [19, 20].

Because the system does not lend itself to a mean-field analysis, I try another approach. If t_1 were equal to τ the system would not deviate from independence and the only effective term of the Hamiltonian in equation (27) in the partition function would be

$$H_0 = -\frac{\mu}{2\tau M} \left(\sum_i \tau_i \right)^2. \quad (28)$$

Such a reduced Hamiltonian is easily treatable with mean-field techniques. Thus, I perform a perturbative expansion for $t_1 \simeq \tau$. Defining $\epsilon = t_1 - \tau$ as the small parameter, the constants

[†] The *characteristics* of a Markov RF are the conditional probabilities of single nodes given the activities of the neighbours; as, e.g., equations (24) and (26).

entering the ‘perturbed’ Hamiltonian are

$$\begin{aligned} t_1 &= \tau + \epsilon(1 - \tau) \\ t_0 &= \tau - \epsilon\tau \\ \beta &= \frac{\epsilon}{\tau} + O(\epsilon^2) \\ \beta\lambda &= \frac{\epsilon}{1 - \tau} + O(\epsilon^2). \end{aligned} \quad (29)$$

Writing H as $H_0 + H_1$ and

$$Z_0 = \sum_{\{\tau_i\}} \exp(-\beta H_0(\{\tau_i\})) \quad (30)$$

up to the second order in ϵ I have

$$Z \simeq Z_0[1 - \langle \beta H_1 \rangle_0 + \frac{1}{2} \langle (\beta H_1)^2 \rangle_0] \quad (31)$$

where $\langle A \rangle_0$ is the average of A in the ensemble with Hamiltonian H_0 (‘unperturbed’ system).

Then, the free energy is (up to $O(\epsilon^3)$)

$$\langle \ln Z \rangle \simeq \ln Z_0 - \langle \langle \beta H_1 \rangle_0 \rangle + \frac{1}{2} \langle \langle [(\beta H_1)^2]_0 - \langle (\beta H_1)_0^2 \rangle \rangle \rangle \quad (32)$$

where $\langle \langle \cdot \rangle \rangle$ indicates the average over the quenched variables.

To verify if the statistics of the system have the desired marginals, I define the ‘observables’

$$\begin{aligned} \tilde{t}_1 &= \frac{1}{\tau M s'} \sum_{i,j} s_{ij} \tau_i \tau_j \\ \tilde{\tau} &= \frac{1}{M} \sum_i \tau_i. \end{aligned} \quad (33)$$

The second one ($\tilde{\tau}$) resembles the expression of the average magnetization typical of the studies on spin systems, while \tilde{t}_1 looks like the energy density of extremely diluted ferromagnets. In the present case, $\tilde{\tau}$ and \tilde{t}_1 are used to extract information about, respectively, the average activity across the system, and the average correlation in pairs of connected sites.

The averages of the quantities in equations (33) can be derived from equation (32), and are respectively equal to $t_1 + O(\epsilon^2)$ and $\tau + O(\epsilon^2)$. This means that the canonical model, in the approximation used in the mathematical treatment, obeys a PDF whose marginals are those of equations (8).

This result strongly supports the belief that the marginal problem analysed in this work (with s' finite) has at least one solution and, consequently, that the Ansatz for the marginal used to solve the original neurobiological problem constitutes a meaningful model. Unfortunately, the perturbative expansion up to order ϵ^2 does not provide an upper bound to t_1 . As already stated, any increase in s' makes the system more strongly correlated; this should reduce the upper bound to t_1 monotonically until, for s' large enough, the statistics should not appreciably deviate from independence ($t_1 = \tau$).

The master equation approach to the averages as in [21, 22] seems to be unsuitable here due to the symmetry of the connections ($s_{mn} = s_{nm}$).

5.2. Analogy with the Ising ferromagnet

One can think about the correlated system in question as an Ising spin model in which each spin interacts ferromagnetically with a small number (about s') of other spins randomly chosen at the beginning. Let us define a ‘cluster centred around a spin’ as the set of all the spins connected to that one; this relation is quenched. It can be shown, at least in an approximate calculation (e.g.

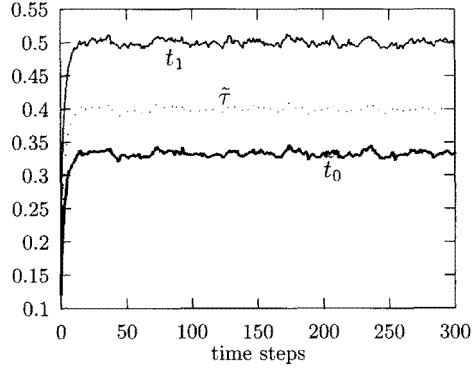


Figure 2. Typical output of a simulation with \tilde{t}_1 and τ in the useful region ($s' = 5$, $\tau = 0.4$, $t_1 = 0.5$, $M = 40\,000$).

large temperature), that, as intuition suggests, the spins of a cluster tend to be oriented parallel to the centre of that cluster even if the temperature is well above the Curie point and the global magnetization is zero. In a sense this phenomenon recalls Weiss domains but in the present model there is no topology: the spins belonging to the same cluster may be scattered throughout the system and may interact with several different clusters. If s is finite, the ferromagnetic model is purely mean-field and the phenomenon of cluster polarization disappears (s_{ij} are homogeneously distributed).

5.3. Simulation results

I have implemented the dynamics in numerical simulations, where s' has been taken between 2 and 8 since this appears to be the interesting region to study.

The main quantities observed in the simulations have been \tilde{t}_1 , $\tilde{\tau}$, \tilde{t}_0 . According to these ‘order parameters’ (actually, the last one is obtainable from the other two), the system soon reaches equilibrium (if the selected parameters t_1 and τ are in due ranges; t_0 is then fixed by relation (15)). The oscillations of the order parameters derive from the finiteness of the simulated net and are in good agreement with the prediction of the variances (see appendix B).

In figure 2 a typical output of the simulation when t_1 is in the useful range (see below) is shown. At first the dynamics drive the system activities toward the equilibrium distribution. Then, the order parameters oscillate around their averages: $\tilde{t}_1 \simeq 0.5000 \pm 0.0043$, $\tilde{\tau} \simeq 0.4001 \pm 0.0037$, $\tilde{t}_0 \simeq 0.3335 \pm 0.0038$, with standard deviations that are compatible with the estimates based on the pattern statistics (appendix B) and that decrease as the number of nodes increases. The relation in equation (15) is fully respected, that is:

$$(1 - \tilde{\tau})\tilde{t}_0 = \tilde{\tau}(1 - \tilde{t}_1). \quad (34)$$

Thus, in this case, the marginals family of equations (8) is consistent.

Figure 3 shows the output of a simulation with parameters t_1 and τ biologically plausible. The order parameters assume the following values: $\tilde{t}_1 \simeq 0.3956 \pm 0.0100$, $\tilde{\tau} \simeq 0.0100 \pm 0.0002$, $\tilde{t}_0 \simeq 0.0060 \pm 0.0002$. The standard deviations are compatible with the estimates and the relation in equation (15) is fully respected. Also, in this case, equations (8) are consistent.

In these conditions the system does not switch among different equilibrium values of the order parameters, possibly indicating that the dynamics cannot switch to different metastable states, at least at the level of the measured quantities. This has been verified by testing the system dynamics for a very long time and for several quenched structures of connections.

The other important result from simulations is the estimation of the upper bound t_1^{\max} to

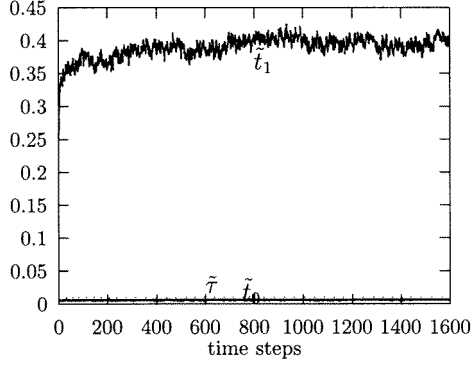


Figure 3. Output of a simulation with biologically plausible parameters: $t_1 = 0.4$, $\tau = 0.01$, $s' = 4$, $M = 10^5$. The quantities $\tilde{\tau}$ and \tilde{t}_0 are very small compared with \tilde{t}_1 ($\frac{t_1}{\tau} = 40$).

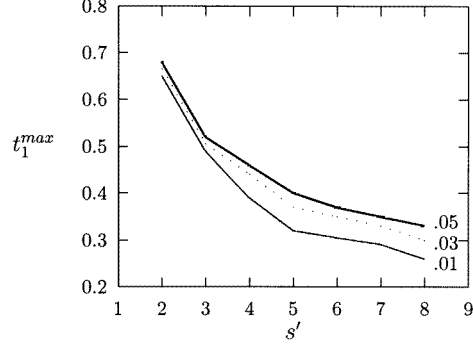


Figure 4. Estimation, from simulations, of the upper bounds to t_1 as functions of s' , for three values of τ (0.01, 0.03, 0.05).

t_1 : that is, the value of t_1 beyond which the system is no longer able to sustain the distribution with the desired marginals. If t_1 is increased above its upper bound, the average values of \tilde{t}_1 , $\tilde{\tau}$, \tilde{t}_0 are no longer equal to t_1 , τ , t_0 , being \tilde{t}_1 and $\tilde{\tau}$ lower than t_1 and τ , respectively. In this out-region case the relation in equation (15) is not respected.

Figure 4 shows the dependence of t_1^{\max} on s' for three different values of τ (around the biologically interesting values). It confirms that the larger s' , the smaller the useful range available to t_1 . This result, obviously, could be non-universal; other dynamics, that is other Ansatz, on the probability of equation (5), might provide the system with a larger useful range for t_1 . So, the bound depicted in figure 4 might not necessarily be the one implied by the general assumption about the marginals in equation (7). Further investigation is going on to find an analytical prediction of the bound and to verify its generality.

Simulations also show the necessity of the structural factors, at least when the RF is defined by equation (26). For example, it is very clear from computer outputs that the correlation between two unconnected nodes that share a common neighbour is, for many such pairs of nodes, appreciably larger than in the case of independence (e.g. \dagger , $\langle \tau_m \tau_n \rangle = 0.0055 \pm 0.0007$ instead of 0.0025 ± 0.0005 , being $\tau = 0.05$, $t_1 = 0.3$, $s' = 4$, $M = 20\,000$; or $\langle \tau_m \tau_n \rangle = 0.0044 \pm 0.0005$ instead of 0.0025 ± 0.0005 , being $\tau = 0.05$, $t_1 = 0.2$, $s' = 4$, $M = 100\,000$). The values are in good agreement with the value $\tau t_1^2 + (1 - \tau)t_0^2$ from equation (23), which is what one would expect from a generating dynamics such as the one implemented here considering that the probability for a pair of nodes to have more than one common neighbour in the diluted graph is negligible, and that the contribution given by longer connecting paths is not dominant. The ensemble average of the activity of any node is in good agreement with the fixed value of τ . The correlation of the activities of connected nodes is usually in excellent agreement with the expected value $t_1 \tau$.

6. Conclusions

In the first part of this paper I have proved a necessary and sufficient condition for the consistency of a class of families of marginal distributions defined on finite sets of discrete

\dagger The indicated standard deviations are not the experimental ones (overestimates) but those inferred from a binomial distribution over trials with the mean equal to the indicated experimental mean.

random variables. I have shown that a family of marginals is consistent if and only if such marginals equal the corresponding marginalizations of a Boltzmann–Gibbs distribution constructed according to a precise rule.

The second part of the paper has been dedicated to the study of a particular system from theoretical neuroscience. In a recent work [14] it has been shown that one of the ingredients to construct a plausible model of the inhomogeneous neuronal networks subserving memory retrieval in the associative areas of primate neocortex is the introduction of correlations among the activities of connected modules of recurrent networks. Modules, consisting of densely connected neurons, can each be in one of two possible states (either active or non-active). Any two modules are more likely to be in the same state of activity if connected by a channel of communication through which, as an abstraction, neuronal projections are allowed to pass. This biological system is modelled as a set of binary random variables (corresponding to modular activities) that are randomly correlated, thus constituting a graph where the presence of an edge between two nodes is to be interpreted as the existence of prominent interdependence between the two corresponding RVs.

The positive results obtained in [14] are valid only if one verifies the hypothesis used there about the marginal distributions of the activities of pairs of connected and unconnected units (modules). In particular, the set of marginals has to be confirmed as a consistent family.

Already existing (to my knowledge) theorems about probability distributions with given marginals cannot be applied because of the large number of RVs together with the randomness in the structure of correlations.

I have formulated the problem in terms of probability distributions on random graphs. I have demonstrated that the stochastic system cannot be realized if any node is connected with finite probability to any other node while the global average activity is fixed. Then, to make use of the condition derived in the first part of the paper, I have produced a fictitious dynamics whose asymptotic distribution has the desired marginals, thus showing that the set of given marginals is consistent. For small correlations I have analytically accounted for the equilibrium properties of the system. For arbitrary correlations I have performed numerical simulations whose outputs confirm (within the statistical fluctuations) the analytical results more generally and provide the upper bound to the node activity correlation as a function of the average number of neighbours per node, for some values of the average activity. Both the approaches strongly suggest that the particular marginals family is consistent, thus supporting the underlying neuronal theory. An analogy with a magnetic system is also proposed to emphasize the main phenomenon.

Further investigations are going on to find out an analytical relation between the upper bound to the correlation and the number of neighbours that fits the numerical results.

Appendix A. Connectivity of the random graph

Let p be the probability for any node, different from node m , to be connected to node m through a path belonging to the graph. Then

$$\begin{aligned} 1 - p &= \sum_{k=0}^{M-1} \binom{M-1}{k} s^k (1-s)^{M-1-k} (1-p)^k \\ &= \left(1 - \frac{ps'}{M-1}\right)^{M-1} \rightarrow e^{-ps'} \end{aligned} \quad (\text{A.1})$$

in the thermodynamic limit. It follows that

$$s' = -\frac{\ln(1-p)}{p}. \quad (\text{A.2})$$

Appendix B. Self-averaging

As already stated, the average over the patterns of the average activity in the network is

$$\left\langle \frac{1}{M} \sum_m \tau_m \right\rangle = \tau. \quad (\text{B.1})$$

The fluctuation around this mean can be calculated:

$$\begin{aligned} \sigma^2 &= \left\langle \frac{1}{M} \sum_m (\tau_m - \tau) \cdot \frac{1}{M} \sum_n (\tau_n - \tau) \right\rangle \\ &= \frac{1}{M^2} \sum_{m,n} \langle (\tau_m - \tau)(\tau_n - \tau) \rangle \\ &= \frac{\tau(1-\tau)}{M} + \frac{2}{M^2} \sum_{(m,n)} \langle \tau_m \tau_n \rangle - \frac{4\tau}{M^2} \sum_{m,n} \langle \tau_m \rangle + \frac{M-1}{M} \tau^2 \\ &= \frac{\tau(1-\tau)}{M} \frac{2}{M^2} \sum_{(m,n)} [f_{mn}^{11} s_{mn} t_1 \tau + h_{mn}^{11} (1-s_{mn}) x \tau] \\ &\quad - \frac{2(M-1)\tau}{M^2} \sum_m \langle \tau_m \rangle + \frac{M-1}{M} \tau^2. \end{aligned} \quad (\text{B.2})$$

Using (9):

$$\sigma^2 = \frac{\tau(1-\tau)}{M} + \frac{M-1}{M} s t_1 \tau + \frac{M-1}{M} (1-s)x\tau - \frac{M-1}{M} \tau^2. \quad (\text{B.3})$$

In the thermodynamic limit ($M \rightarrow \infty$):

$$\sigma^2 \longrightarrow s\tau(t_1 - x) + \tau(x - \tau). \quad (\text{B.4})$$

Assuming self-averaging means to assume that the right-hand side of equation (B.4) vanishes. This implies that

$$x = \frac{\tau - s t_1}{1 - s} \quad (\text{B.5})$$

which is easily seen to be not larger than τ .

In the simulations I have taken $s = \frac{s'}{M-1}$, with s' finite, and, according to equation (B.5), $x = \tau$. Thus, the fluctuation is given by

$$\sigma^2 = \frac{\tau(1-\tau)}{M} + \frac{M-1}{M} s\tau(t_1 - \tau). \quad (\text{B.6})$$

In a very similar way, fluctuation of the parameter \tilde{t}_1 can be estimated:

$$\langle \tilde{t}_1 \rangle = t_1 \quad (\text{B.7})$$

$$\langle (\tilde{t}_1 - t_1)^2 \rangle \simeq \frac{2}{M} \frac{t_1}{\tau} \left(t_1 + \frac{1}{s'} \right) \quad (\text{B.8})$$

having assumed the approximation

$$\begin{aligned} \mathcal{P}(\tau_i = 1, \tau_j = 1, \tau_k = 1, \tau_l = 1 | s_{mn} = 1, s_{kl} = 1) \\ \simeq \mathcal{P}(\tau_i = 1, \tau_j = 1 | s_{mn} = 1) \cdot \mathcal{P}(\tau_k = 1, \tau_l = 1 | s_{kl} = 1) \end{aligned} \quad (\text{B.9})$$

with i, j, k, l being four different site indices. The estimate in (B.6) is in excellent agreement with simulation data, while that in (B.8) is a slight overestimate, perhaps due to the approximation (B.9).

References

- [1] Vorobev N 1962 *Theory Prob. Appl.* **7** 147
- [2] Kellerer H 1964 *Z. Wahrscheinlichkeitstheorie* **3** 247
- [3] Aleksandrov P 1956 *Combinatorial Topology* (Baltimore, MD: Graylock Press)
- [4] Kemeny J, Snell J, Knapp A and Griffieath D 1976 *Denumerable Markov Chains* 2nd edn (New York: Springer) ch 10
- [5] Braitenberg V and Shutz A 1991 *Anatomy of the Cortex: Statistics and Geometry* (Berlin: Springer)
- [6] Hopfield J J 1982 *Proc. Natl Acad. Sci., USA* **79** 2554
- [7] Mountcastle V B 1997 *Brain* **120** 701
- [8] Miyashita Y and Chang H 1988 *Nature* **331** 68
- [9] Fuster J M 1997 *Trends Neurosci.* **20** 451
- [10] Tanaka K 1996 *Neural Netw.* **9** 1459
- [11] Amit D, Brunel N and Tsodyks M 1994 *J. Neurosci.* **14** 6435
- [12] O'Kane D and Treves A 1992 *J. Phys. A: Math. Gen.* **25** 5055
- [13] O'Kane D and Treves A 1992 *Network* **3** 379
- [14] Fulvi Mari C and Treves A 1998 *Biosystems* **48** 47
- [15] Kellerer H 1991 *Advances in Probability Distributions with Given Marginals* ed G Dall'Aglio *et al* (Dordrecht: Kluwer)
- [16] Jouve B, Rosenstiehl P and Imbert M 1998 *Cereb. Cortex* **8** 28
- [17] Binder K and Heermann D W 1997 *Monte Carlo Simulation in Statistical Physics* (Berlin: Springer)
- [18] Hammersley J and Clifford P 1971 unpublished
Clifford P 1990 *Disorder in Physical Systems. Volume in Honour of J M Hammersley 70th Birthday* ed G Grimmett and D Welsh (Oxford: Clarendon)
- [19] Viana L and Bray A 1985 *J. Phys. C: Solid State Phys.* **18** 3037
- [20] Kanter I and Sompolinsky H 1987 *Phys. Rev. Lett.* **58** 164
- [21] Derrida B, Gardner E and Zippelius A 1987 *Europhys. Lett.* **4** 167
- [22] Barkai E, Kanter I and Sompolinsky H 1990 *Phys. Rev. A* **41** 590